

## LOWER BOUNDS FOR MULTIDIMENSIONAL ZERO SUMS

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Let  $f(n, d)$  denote the least integer such that any choice of  $f(n, d)$  elements in  $\mathbb{Z}_n^d$  contains a subset of size  $n$  whose sum is zero. Harborth proved that  $(n-1)2^d + 1 \leq f(n, d) \leq (n-1)n^d + 1$ . The upper bound was improved by Alon and Dubiner to  $c_d n$ . It is known that  $f(n, 1) = 2n - 1$  and Reiher proved that  $f(n, 2) = 4n - 3$ . Only for  $n = 3$  it was known that  $f(n, d) > (n-1)2^d + 1$ , so that it seemed possible that for a fixed dimension, but a sufficiently large prime  $p$ , the lower bound might determine the true value of  $f(p, d)$ . In this note we show that this is not the case. In fact, for all odd  $n \geq 3$  and  $d \geq 3$  we show that  $f(n, d) \geq 1.125^{\lfloor \frac{d}{3} \rfloor} (n-1)2^d + 1$ .

**1. Introduction**

A classical result of Erdős, Ginzburg, and Ziv [8] states that amongst any  $2n - 1$  integers one can choose  $n$  such that their sum is divisible by  $n$ .

Harborth [14] considered the corresponding problem for  $d$ -dimensional integer lattices. Let  $f(n, d)$  denote the minimal number such that any choice of  $f(n, d)$  not necessarily distinct vectors  $v_i \in \mathbb{Z}_n^d$  contains a subset of  $n$  vectors whose sum is  $0 \in \mathbb{Z}_n^d$ . Harborth proved that

$$(n-1)2^d + 1 \leq f(n, d) \leq (n-1)n^d + 1.$$

The lower bound follows from the example in which there are  $n - 1$  copies of each of the  $2^d$  vectors with entries 0 or 1. The upper bound follows since any set of  $(n-1)n^d + 1$  vectors must contain, by the pigeonhole principle,  $n$  vectors which are equivalent modulo  $n$ .

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The upper bound was greatly improved upon by Alon and Dubiner [2]:

$$f(n, d) \leq c_d n.$$

For composite integers  $n = n_1 n_2$  an upper bound on  $f(n, d)$  can be derived by bounds on  $f(n_1, d)$  and  $f(n_2, d)$  as follows (see Harborth [14]):

$$(1) \quad f(n_1 n_2, d) \leq \min \{f(n_1, d) + (f(n_2, d) - 1)n_1, f(n_2, d) + (f(n_1, d) - 1)n_2\}.$$

One therefore often restricts the consideration to a prime argument  $n = p$ .

For  $d = 2$  it was conjectured that  $f(n, 2) = 4n - 3$ , so that the lower bound would determine the correct value. In fact it suffices to prove this conjecture for prime values of  $n$ , since the conjecture then follows by (1). There were partial results in favour of this conjecture, see Kemnitz [16], as well as Gao [10], [11], [12], and Thangadurai [21].

Rónyai [20] proved that for primes  $p$  one has  $f(p, 2) \leq 4p - 2$ , which implies that  $f(n, 2) \leq 4.1n$ . Gao [13] extended this to powers of primes:  $f(p^a, 2) \leq 4p^a - 2$ . Reiher [19] eventually proved this conjecture:  $f(n, 2) = 4n - 3$ .

For  $d \geq 3$  very little is known.

$$\begin{aligned} f(2^a, d) &= (2^a - 1)2^d + 1 && \text{(see Harborth [14]),} \\ f(3, 3) &= 19 && \text{(see Harborth [14], Brenner [3]),} \\ f(3, 4) &= 41 && \text{(see Pellegrino [18], Brown and} \\ &&& \text{Buhler [4], Brenner [3], Kemnitz [16]),} \\ 91 \leq f(3, 5) &\leq 121 && \text{(see Kemnitz [15]),} \\ f(3, 5) &= 91 && \text{(see Edel et al. [7], [5])} \\ 225 \leq f(3, 6) &\leq 229 && \text{(see Edel et al. [7], [5])} \\ f(3, 18) &\geq 300 \times 2^{12} && \text{(see Frankl, Graham, Rödl [9]),} \\ f(3, d) &\geq 2.179^d \text{ for } d \geq d' && \text{(see Frankl, Graham, Rödl [9]),} \\ f(3, d) &\geq 2.217389^d \text{ for } d \geq d' && \text{(see Edel [6])} \\ f(n, d) &= o(n^d) \text{ for fixed } n, \text{ as } d \rightarrow \infty && \text{(see Alon, Dubiner [2])} \end{aligned}$$

where  $d'$  is sufficiently large. The result  $f(3, d) = o(3^d)$  (Brown and Buhler [4], and Frankl, Graham and Rödl [9]) can be derived from the Szemerédi–Roth theorem on arithmetic progressions of length 3. Ruzsa proved that  $f(3, d) = O\left(\frac{3^d}{\sqrt{d}}\right)$  holds (see Meshulam [17]) and Meshulam proved a more general result on arithmetic progressions of length three in finite abelian groups which implies as a special case  $f(3, d) \leq 2\frac{3^d}{d}$  (see also [17]).

The exact determination of  $f(n, d)$  is a very difficult problem. Harborth's lower bound has been improved only in the above mentioned very few special cases with  $n = 3$ . Not a single case for odd  $n > 3$  was known with  $f(n, d) > 2^d(n - 1) + 1$ . It seemed conceivable that for any fixed dimension and a

sufficiently large prime  $p$  the lower bound determines the correct value of  $f(p, d)$ . In this note we show that this is not the case. We prove the following theorem:

**Theorem.** *Let  $n \geq 3$  be an odd integer. The following inequality holds:*

$$f(n, d) \geq 1.125 \lfloor \frac{d}{3} \rfloor (n-1)2^d + 1.$$

Formally, the bound is also valid for  $d=1$  and  $d=2$ , but is not new in these cases. This theorem has a number of simple corollaries:

**Corollary 1.** *For fixed  $d$  this implies that the constant  $c_d$  in Alon and Dubiner's result must satisfy*

$$c_d \geq 2^d 1.125 \lfloor \frac{d}{3} \rfloor.$$

If we take a fixed  $n$  and a large dimension  $d$ , then this implies:

**Corollary 2.** *For odd  $n$  and sufficiently large  $d \geq d'$  the following lower bound holds:*

$$f(n, d) \geq 2.08^d.$$

For  $n=3$  this is weaker than the existing bound  $f(3, d) \geq 2.217389^d$  (for large  $d$ ).

It is interesting to note that we have for the general case considered in the [Theorem](#) a simple and uniform proof based on only one particular value:  $f(3, 3) = 19$ . It is not surprising that this can be extended to  $f(3, d)$ , but it is certainly surprising that the proof can be extended to different  $n$ .

For the related problem, where  $g(n, d)$  denotes the minimal number such that any set of  $g(n, d)$  *distinct* vectors in  $\mathbb{Z}_n^d$  has a zero sum of length  $n$ , we show similar bounds. It is known that  $g(n, d) \geq (n-1)2^{d-1} + 1$  for  $n \geq 3$ , and  $g(n, d) \geq n2^{d-1} + 1$  for even  $n$ ,  $g(3, 3) = 10$ ,  $g(3, 4) = 21$ , (see Harborth [14], Brenner [3], Frankl, Graham and Rödl [9], and Kemnitz [15], [16]). By the pigeonhole principle we have that  $g(n, d) \geq \frac{f(n, d)-1}{n-1} + 1$ . Therefore the theorem immediately implies the following

**Corollary 3.** *Let  $n \geq 3$  be an odd integer. The following bound holds:*

$$g(n, d) \geq 1.125 \lfloor \frac{d}{3} \rfloor 2^d + 1.$$

*In particular, for large  $d$ :*

$$g(n, d) \geq 2.08^d.$$

For fixed  $d$  and large  $n$  the corollary is much weaker than the [theorem](#). One trivially has that  $n \leq g(n, d)$ . The lower bound of the corollary is independent of  $n$  and therefore in this respect not optimal. But for fixed  $n$  and large  $d$  the corollary above may be of interest.

## 2. Proof

In the proof of the theorem we start off from Harborth's example which shows that  $f(3, 3) \geq 19$ . We first extend this in dimension 3 from  $n = 3$  to arbitrary  $n$ . We then increase the dimension by an explicit product construction.

We take the 9 vectors considered by Harborth but we consider these in  $\mathbb{Z}_n^3$  (not only in  $\mathbb{Z}_3^3$ ).

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

If  $n \geq 3$ , then these vectors are distinct in  $\mathbb{Z}_n^3$ . We shall prove:

**Lemma.** *Let  $n \geq 3$  be an odd integer. If any  $n$  vectors taken from a multiset of the above 9 vectors add to  $0 \in \mathbb{Z}_n^3$ , then necessarily one has taken  $n$  times the very same vector.*

To avoid such kind of zero sum our example in  $d=3$  starts off with  $n-1$  copies of each of the above 9 elements in  $\mathbb{Z}_n^3$ . In dimension  $d > 3$  we construct our vectors as follows. Up to dimension  $3 \lfloor \frac{d}{3} \rfloor$  we take all  $9^{\lfloor \frac{d}{3} \rfloor}$  vectors which are composed out of any of the 9 vectors in the coordinates  $3i+1$  up to  $3i+3$ , where  $i=0, \dots, \lfloor \frac{d-3}{3} \rfloor$ . In the remaining  $d_0=0, 1$  or  $2$  coordinates we take all  $2^{d_0}$  combinations of vectors composed of 0 and 1 only. Note that  $2^{d_0} \times 2^{3 \lfloor \frac{d}{3} \rfloor} = 2^d$ . We take any of these vectors  $n-1$  times such that the total number of used vectors is  $1.125^{\lfloor \frac{d}{3} \rfloor} 2^d (n-1)$ .

Suppose that there is a choice of  $n$  vectors which sums to the zero vector, modulo  $n$ . Any choice of  $n$  vectors which sums in the last  $d_0$  coordinates to 0 or  $n$  must necessarily have  $n$  times the same value of  $\varepsilon_i \in \{0, 1\}$ . Moreover, we use the lemma for triples of dimensions  $(3i+1, 3i+2, 3i+3)$ , where  $i=0, \dots, \lfloor \frac{d}{3} \rfloor$ . Suppose for a moment the lemma is proven: then a zero sum must take in these three coordinates  $n$  copies of the same entries. But since there are  $n-1$  copies of each  $d$ -dimensional vector only, the  $n$  vectors cannot be identical. They differ in at least one of the three dimensions  $3i+1$  up to

$3i+3$  (say) or in the last  $d_0$  dimensions. In these dimensions there is no zero sum of length  $n$ , which proves the [theorem](#).

It is therefore sufficient to prove the lemma. Suppose our choice of the  $n$  vectors uses  $a_i$  copies of the  $i$ -th vector, where  $i = 1, \dots, 9$  and where  $0 \leq a_i \leq n-1$ . We then have the following system of equations.

$$\begin{aligned} (1) \quad & a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = n \\ (2) \quad & 2a_1 + a_6 + a_7 + a_8 + a_9 \equiv 0 \pmod{n} \\ (3) \quad & a_1 + a_4 + a_5 + a_8 + 2a_9 \equiv 0 \pmod{n} \\ (4) \quad & 2a_1 + a_3 + a_5 + a_7 + 2a_8 + 2a_9 \equiv 0 \pmod{n}. \end{aligned}$$

We show first that the three expressions on the left hand side of (2), (3) and (4) must be equal to  $n$ . In order to see this, we must consider various cases.

**Case 1)** In equation (2) we must exclude that  $2a_1 + a_6 + a_7 + a_8 + a_9 = 0$  or  $\geq 2n$ . Similarly, cases 2) and 3) will treat the equations (3) and (4).

**Case 1a)** Equation (1) and  $0 \leq a_i < n$  shows that  $2a_1 + a_6 + a_7 + a_8 + a_9 < 2n$ .

**Case 1b)** So let us suppose for a contradiction that  $2a_1 + a_6 + a_7 + a_8 + a_9 = 0$ . This implies that  $a_1 = a_6 = a_7 = a_8 = a_9 = 0$ . Our system of equations simplifies to

$$\begin{aligned} (1') \quad & a_2 + a_3 + a_4 + a_5 = n \\ (3') \quad & a_4 + a_5 \equiv 0 \pmod{n} \\ (4') \quad & a_3 + a_5 \equiv 0 \pmod{n}. \end{aligned}$$

Moreover

$$\begin{aligned} (1') - (3') \quad & a_2 + a_3 \equiv 0 \pmod{n} \\ (1') - (4') \quad & a_2 + a_4 \equiv 0 \pmod{n}. \end{aligned}$$

If we assume that  $a_4 + a_5 = n$ , then  $a_2 = a_3 = 0$ , by (1'). But then by (1') - (4'):  $a_4 \equiv 0 \pmod{n}$ . Then  $0 \leq a_i < n$  shows that  $a_4 = 0$ , so that  $a_5 = n$ , a contradiction to  $a_5 < n$ . Similarly, if we assume that  $a_2 + a_3 = n$ , so that  $a_4 = a_5 = 0$ , then the same kind of contradiction follows from (1') - (4'):  $a_2 \equiv 0 \pmod{n}$ , and  $a_3 = n$ .

**Case 2a)** Since  $a_9 < n$  we can exclude that

$$(3) \quad a_1 + a_4 + a_5 + a_8 + 2a_9 \geq 2n.$$

**Case 2b)** So, let us suppose that

$$(3) \quad a_1 + a_4 + a_5 + a_8 + 2a_9 = 0,$$

which implies that  $a_1 = a_4 = a_5 = a_8 = a_9 = 0$ .

Our system of equations then is

$$\begin{aligned} (1') \quad & a_2 + a_3 + a_6 + a_7 = n \\ (2') \quad & a_6 + a_7 = n \quad \text{by case 1)} \\ (4') \quad & a_3 + a_7 \equiv 0 \pmod n. \end{aligned}$$

This implies  $a_2 = a_3 = 0$ . Then (4') implies that  $a_7 \equiv 0 \pmod n$ , i.e.  $a_7 = 0$ , which contradicts  $a_6 < n$ .

**Case 3a)** Suppose that

$$(4) \quad 2a_1 + a_3 + a_5 + a_7 + 2a_8 + 2a_9 \geq 2n.$$

Then  $2(1) - (4)$  gives

$$2a_2 + a_3 + 2a_4 + a_5 + 2a_6 + a_7 \leq 0,$$

so that

$$a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0.$$

Our system of equations then is:

$$\begin{aligned} (1') \quad & a_1 + a_8 + a_9 = n \\ (2') \quad & 2a_1 + a_8 + a_9 = n \quad \text{by case 1)} \\ (3') \quad & a_1 + a_8 + 2a_9 = n \quad \text{by case 2)}. \end{aligned}$$

This implies that  $a_1 = a_9 = 0$ ,  $a_8 = n$ , which contradicts  $a_8 < n$ .

**Case 3b)** So, we assume that

$$(4) \quad 2a_1 + a_3 + a_5 + a_7 + 2a_8 + 2a_9 = 0,$$

which implies

$$a_1 = a_3 = a_5 = a_7 = a_8 = a_9 = 0.$$

By equation (2) we see that  $a_6 = n$ , which contradicts  $a_6 < n$ .

The above case study shows that the left hand sides of (1), (2), (3), (4) are all equal to  $n$ . Let us look at equation

$$(1) + (4) - (2) - (3) \quad a_2 + 2a_3 + a_5 + a_7 + a_8 = 0.$$

Because of  $0 \leq a_i$  we find that

$$a_2 = a_3 = a_5 = a_7 = a_8 = 0.$$

Therefore equation (4) simplifies to

$$2a_1 + 2a_9 = n,$$

which is a contradiction, since  $n$  is odd, by assumption. ■

**Remarks.** It seems conceivable that starting off from other examples for small fixed  $n$  and  $d$ , one might be able to improve the lower bound. But it is not at all obvious that this will work for any particular value of  $f(n, d)$ . It is indeed somewhat surprising that an argument modulo 3 can be extended to an argument modulo all odd integers.

For even  $n$  the lower bound cannot hold in general since for  $n = 2^a$  one knows that  $f(2^a, d) = (2^a - 1)2^d + 1$ .

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